1 Derivation Details of Static Equilibrium Problems

The derivation of ANM formulas for solving the static equilibrium problem is similar to the inverse problem presented in Appendix A. The main differences are that (a) we expand the position $x$ and $\lambda$ in deformed space as

$$\begin{bmatrix} x(a) \\ \lambda(a) \end{bmatrix} = \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix} + \sum_{k=1}^{n} (a - a_0)^k \begin{bmatrix} x_k \\ \lambda_k \end{bmatrix},$$  \hspace{1cm} (1)$$

and (b) we express the internal force using the first Piola-Kirchhoff stress tensor, $P$. We therefore introduce auxiliary variables to establish quadratic relationships between $P$ and $x$:

$$\begin{align*}
P &= FS \\
S &= \mu s^{(2)} I - \frac{\mu}{3} J^i C^{inv} + \kappa J^i C^{inv} \\
J^i &= s^{(2)} I_c \\
s^{(2)} &= s^{(1)} s^{(1)} \\
J^{inv} &= s^{(2)} s^{(1)} \\
J J^{inv} &= 1 \\
J^j &= (J - 1) J \\
J &= \epsilon_{lmn} \hat{F}_{lm} F_{n3} \\
\hat{F}_{lm} &= F_{l1} F_{m2} \\
CC^{inv} &= I \\
I_c &= I : C \\
C &= F^T F \\
F &= [x] [X]^{inv}
\end{align*}$$

Here the notations are the same as used in Appendix A.
Next, we derive the expansions for all the auxiliary variables. For a given order $k$ ($1 \leq k \leq N$), we have

\[
\begin{align*}
    P_k &= F_0 S_k + F_k S_0 + \sum_{r=1}^{k-1} F_r S_{k-r} \\
    S_k &= \mu s_k^{(2)} I - \frac{\mu}{3} \left( J_0^I C_k^{inv} + J_k^I C_0^{inv} + \sum_{r=1}^{k-1} J_r^I C_{k-r}^{inv} \right) \\
        &\quad + \kappa \left( J_0^J C_k^{inv} + J_k^J C_0^{inv} + \sum_{r=1}^{k-1} J_r^J C_{k-r}^{inv} \right) \\
    J_k^I &= s_0^{(2)} (I_c)_k + s_k^{(2)} (I_c)_0 + \sum_{r=1}^{k-1} s_r^{(2)} (I_c)_{k-r} \\
    s_k^{(2)} &= s_0^{(1)} s_k^{(1)} + \sum_{r=1}^{k-1} s_r^{(1)} s_{k-r} = 2s_0^{(1)} s_k^{(1)} + \sum_{r=1}^{k-1} s_r^{(1)} s_{k-r} \\
    J_k^{inv} &= -J_0^{inv} J_k^{inv} - J_0^{inv} \sum_{r=1}^{k-1} J_r^{inv} J_{k-r} \\
    J_k^J &= 2J_0 J_k + \sum_{r=1}^{k-1} J_r J_{k-r} - J_k \\
    J_k &= e_{1mn} \left[ (\tilde{F}_{lm})_k (F_{m3})_0 + (\tilde{F}_{lm})_0 (F_{m3})_k + \sum_{r=1}^{k-1} (\tilde{F}_{lm})_r (F_{m3})_{k-r} \right] \\
    (\tilde{F}_{lm})_k &= (F_{l1})_k (F_{m2})_0 + (F_{l1})_0 (F_{m2})_k + \sum_{r=1}^{k-1} (F_{l1})_r (F_{m2})_{k-r} \\
    C_k^{inv} &= -C_0^{inv} C_k C_0^{inv} - C_0^{inv} \sum_{r=1}^{k-1} C_r C_{k-r}^{inv} \\
    C_k &= F_0^T F_k + F_k^T F_0 + \sum_{r=1}^{k-1} F_r^T F_{k-r} \\
    F_k &= [x]_k [X]^{inv}
\end{align*}
\]

Here $[x]_k$ is the expansion coefficients of $[x]$. Because all elements of $[x]$ is linear with respect to the vector $x$, $[x]_k$ is also linear with respect to $x_k$ in Eq. (1). With these expansions, we follow the same process as described in §4.2 and Appendix A to assemble linear systems for solving $(x_k, \lambda_k)$. The remaining part of the algorithm is the same as the inverse static equilibrium solver presented in the main text of the paper.